概率论与数理统计

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一、填空题

1.

$$E\left(\frac{1}{X}\right) = \int_0^{\infty} 2e^{-x^2}\,dx$$

由正态分布

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}\,dx = 1$$

取

$$\sigma^2 = \frac{1}{2}, \mu = 0$$

得

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi}} e^{-x^2}\,dx = 1$$

$$\therefore E\left(\frac{1}{X}\right) = \int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$$

2.

$$P(B|A) = \frac{P(BA)}{P(A)} = 0.5, \quad P(BA) = 0.2$$

$$P(B|\overline{A}) = \frac{P(B\overline{A})}{P(\overline{A})} = 0.6, \quad P(B\overline{A}) = 0.36$$

$$P(B) = P(BA) + P(B\overline{A}) = 0.56$$

$$P(AC|B) = \frac{P(ACB)}{P(B)} = \frac{P(AB)\cdot P(C)}{P(B)} = \frac{0.2\times 0.6}{0.56} = \frac{12}{56} = \frac{3}{14}$$

3.

$$P(X=0) = e^{-2}, \quad P(X=1) = 2e^{-2}, \quad P(X=2) = 2e^{-2}$$

$$P(X \le 1) = 3e^{-2}$$

$$\begin{cases} X=0, Y=2 & e^{-2}\cdot 2e^{-2} \\ X=1, Y=1 & 2e^{-2}\cdot 2e^{-2} \\ X=2, Y=0 & e^{-2}\cdot 2e^{-2} \end{cases}$$

$$P(X=0|X+Y=2) = \frac{4e^{-4}}{8e^{-4}} = \frac{1}{2}$$

4. (1)

$$\frac{1}{n}(X_1+X_2+\cdots+X_n)\xrightarrow{P}E(X)=\frac{13}{8}$$

(2) 取对数

$$\frac{1}{n}(\ln X_1+\ln X_2+\cdots+\ln X_n)\to E(\ln X)=\frac{1}{2}\ln 2=\ln\sqrt{2}$$

$$\sqrt[n]{X_1X_2\cdots X_n}\xrightarrow{P}e^{\ln\sqrt{2}}=\sqrt{2}$$

(3)

$$P(X>1)=\frac{3}{4},\text{记}Y\text{为}X_1,X_2,\ldots,X_{4800}\text{中大于 1 的个数}$$

$$Y\sim B(4800,\frac{3}{4})\quad\text{大数定律}Y\sim N(3600,900)$$

$$P(Y>3650)=1-\Phi\left(\frac{3650-3600}{30}\right)=1-\Phi\left(\frac{5}{3}\right)$$

5.

$$E(2X-Y-1)=2\times 1-0-1=1$$

$$\mathrm{Var}(2X-Y-1)=4\mathrm{Var}(X)+\mathrm{Var}(Y)-4\mathrm{Cov}(X,Y)=64+25-4\times 0.25\times 9\times 5=69$$

$$2X-Y-1\sim N(1,69)$$

6. (1)

$$\frac{3s^2}{x_\alpha^2(3)}$$

(2)

接受

(3)

$$\overline{X}-X_5-X_6+a\sim N(0,1),\quad a=\mu$$

$$\mathrm{Var}(\overline{X}-X_5-X_6+a)=\frac{\sigma^2}{4}+2\sigma^2=\frac{9}{4}\sigma^2$$

$$\frac{bS}{\frac{3}{2}\sigma}=\sqrt{\frac{3S^2}{3\sigma^2}}=\sqrt{\frac{\chi^2(3)}{3}}=\frac{S}{6}$$

$$b=\frac{3}{2},\quad m=3$$

$$\therefore(\mu,\frac{3}{2},3)$$

(4)

$$\mathrm{Cov}\left[\frac{1}{4}(X_1+X_2+\cdots+X_4),\frac{1}{6}(X_1+X_2+\cdots+X_6)\right]$$

$$=\frac{1}{24}[\mathrm{Var}(X_1)+\mathrm{Var}(X_2)+\cdots+\mathrm{Var}(X_4)]=\frac{1}{24}\times 4\sigma^2=\frac{1}{6}\sigma^2$$

$$\rho=\frac{\frac{1}{6}\sigma^2}{\sqrt{\frac{1}{4}\sigma^2\cdot\frac{1}{6}\sigma^2}}=\frac{2\sqrt{6}}{6}=\frac{\sqrt{6}}{3}$$

二、

(1)

$$\mathrm{Cov}(X_1X_2, X_1X_2X_3) = E(X_1^2X_2^2X_3) - E(X_1X_2)E(X_1X_2X_3) = \frac{1}{8} - \frac{1}{4} \times \frac{1}{8} = \frac{3}{32}$$

X_1X_2	0	1
P	$\frac{3}{4}$	$\frac{1}{4}$

$X_1X_2X_3$	0	1
P	$\frac{7}{8}$	$\frac{1}{8}$

(2)

X_1	X_2	X_3	Y_1	Y_2
0	0	0	0	0
0	0	1	0	0
0	1	0	0	0
0	1	1	0	0
1	0	0	0	0
1	0	1	0	0
1	1	0	1	0
1	1	1	1	1

$Y1\backslash Y2$	0	1
0	$\frac{3}{4}$	0
1	$\frac{1}{8}$	$\frac{1}{8}$

(3) 当 $Y_1 = 1$ 时，

$$P(Y_2 = 0|Y_1 = 1) = \frac{1}{2}$$
$$P(Y_2 = 1|Y_1 = 1) = \frac{1}{2}$$

Y_2	0	1
P	$\frac{1}{2}$	$\frac{1}{2}$

三、

(1) $f_X(x) = F_X'(x) = \begin{cases} ae^x & , x < 0 \\ 3ae^{-3x} & , x \geq 0 \end{cases}$

(2) 利用指数分布均值公式，$a\int_0^{+\infty} 3xe^{-3x}\,dx = \frac{a}{3}$

$$\int_{-\infty}^{0} ae^x\,dx = a\int_{+\infty}^{0} xe^{-x}\,dx = -a\int_0^{+\infty} xe^{-x}\,dx = -a$$
$$\therefore E(X) = -\frac{2}{3}a$$

(3) $F_Y(y) = P(Y \leq y) = P(e^X \leq y) = F(\ln y) = \begin{cases} 0 & , y \leq 0 \\ ay & , 0 < y < 1 \\ b - ay^{-3} & , y \geq 1 \end{cases}$

$$f_Y(y) = F_Y'(y) = \begin{cases} 0 & , y \leq 0 \\ a & , 0 < y < 1 \\ 3ay^{-4} & , y \geq 1 \end{cases}$$

(4) $\begin{cases} F_X(+\infty) = 1 \\ F_X(0^-) = F_X(0^+) \end{cases} \Rightarrow \begin{cases} a = \frac{1}{2} \\ b = 1 \end{cases}$

四、

(1) 当 $x>1$ 时，

$$f_X(x)=\int_x^{+\infty}12y^{-5}\,dy=12\cdot\frac{1}{-5+1}y^{-4}\bigg|_x^{+\infty}=-3y^{-4}\bigg|_x^{+\infty}=-3(0-x^{-4})=3x^{-4}$$

$$f_X(x)=\begin{cases}0, & x\le 1\\ 3x^{-4}, & x>1\end{cases}$$

当 $y>1$ 时，$f_Y(y)=\int_1^y 12y^{-5}\,dx=12y^{-5}(y-1)=12y^{-4}-12y^{-5}$

$$f_Y(y)=\begin{cases}0, & y\le 1\\ 12y^{-4}-12y^{-5}, & y>1\end{cases}$$

不独立

$$P(1<X<1.5)>0,\quad P(Y=1.5)=0$$

$$P(1<X<1.5,Y=1.5)>0$$

$$\therefore P(1<X<1.5,Y=1.5)\ne P(1<x<1.5)\cdot P(Y=1.5)$$

$$\therefore \text{不独立}$$

(2) 当 $y>x$ 时，$f_{X|Y}(x|y)=\frac{f(x,y)}{f_Y(y)}=\frac{1}{y-1}$

$$f_{X|Y}(x|y)=\begin{cases}0, & \text{其他}\\ \frac{1}{y-1}, & y>x>1\end{cases}$$

取 $y=2$，$f_{X|Y}(x|y)=\begin{cases}0, & x\ge 2\\ 1, & 1<x<2\end{cases}$

(3)

$$P(1<X<1.5|Y=2)=\int_1^{1.5}dx=0.5$$

当 $z\le 2$ 时，$F_Z(z)=0$

当 $z>2$ 时，$F_Z(z)=\int_1^{\frac{z}{2}}dx\int_x^{z-x}12y^{-5}\,dy=\int_1^{\frac{z}{2}}\left[-3(z-x)^{-4}+3x^{-4}\right]dx=-16z^{-3}+(z-1)^{-3}+1$

$$\therefore f_Z(z)=F_Z'(z)=\begin{cases}0, & z\le 2\\ 48z^{-4}-3(z-1)^{-4}, & z>2\end{cases}$$

五、

(1) 记 $T=X-\mu$，则

$$f(t)=\begin{cases}2e^{-2t}, & t\ge 0\\ 0, & \text{其他}\end{cases}$$

利用指数分布均值

$$E(T) = \frac{1}{2}$$

$$\therefore E(X) = E(T) + \mu = \frac{1}{2} + \mu, \quad \mu = E(X) - \frac{1}{2}$$

$$\therefore \hat{\mu}_1 = \overline{X} - \frac{1}{2}$$

$$E(\hat{\mu}_1) = E(X - \frac{1}{2}) = E(X) - \frac{1}{2} = \mu$$

$\therefore$ 是无偏估计

(2) $L(x;\mu) = \begin{cases} 2^n e^{-2\sum_{i=1}^{n}(x_i-\mu)}, & x_1, x_2, \ldots, x_n \geq \mu \\ 0, & \text{其他} \end{cases}$

$$\hat{\mu}_2 = \min\{x_1, x_2, \ldots, x_n\}$$

记 $N = \min\{x_1, x_2, \ldots, x_n\}$

$$1 - F_N(t) = [1 - F_X(t)]^n$$

$$F_X(x) = \begin{cases} 1 - e^{-2(x-\mu)}, & x \geq \mu \\ 0, & \text{其他} \end{cases}$$

$$F_N(t) = \begin{cases} 1 - e^{-2n(t-\mu)}, & t \geq \mu \\ 0, & \text{其他} \end{cases}$$

$$f_N(t) = \begin{cases} 2ne^{-2n(t-\mu)}, & t \geq \mu \\ 0, & \text{其他} \end{cases}$$

$$E(N) = \int_{\mu}^{+\infty} 2nte^{-2n(t-\mu)}\,dt = \frac{1}{2n} + \mu$$

$\therefore \hat{\mu}_2 \xrightarrow{P} E(N) = \frac{1}{2n} + \mu$

当 $n \to +\infty$ 时，$\hat{\mu}_2 \xrightarrow{P} \mu$，$\therefore$ 是相合估计

(3) $Mse(\hat{\mu}_1) = Var(\hat{\mu}_1) + [E(\hat{\mu}_1) - \mu]^2 = Var(\overline{X}) = \frac{1}{4n}$

$$Mse(\hat{\mu}_2) = Var(\hat{\mu}_2) + [E(\hat{\mu}_2) - \mu]^2 = \frac{1}{4n^2} + \frac{1}{4n^2} = \frac{1}{2n^2}$$

当 $n \geq 3$ 时，$Mse(\hat{\mu}_2) < Mse(\hat{\mu}_1)$

$\therefore \hat{\mu}_2$ 更有效